March 21, 2016 2:00 PM

We heard your NO
One of our students is trying to get our video and microphone
And onenote draw to work.

Yay! One thing done.

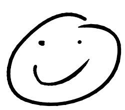
Is your volume on?

Are you sure?

Can you speak?
We can't hear you.

We're restarting
BRB

We can hear you again! Oh. Darn WOOOHOOO



Before dealing with proof of theorem 10.1.1
we consider some "strange" properties of convergence in distribution

Notation: ($X_n \Rightarrow X$ or $F_n \Rightarrow F$)
We say X_n converges in distribution to X iff
$F_n(x) = P(X_n \le x) \to F(x) = P(X \le x)$
these "P" are not necessarily equal
ie X_n, X do not need to be defined on the same prob. triple

Is there some stronger sense that $\Rightarrow$ is equivalent to?
or do we really need Theorem 10.1.1?

Consider X_i iid $E(X_i) = 0$, $Var(X_i) = 1$

$z = \frac{1}{\ }(X_1 + \cdots + X_n)$ that $Z_n \to Z$ a.s. (or in prob?)

$Z_n = \frac{1}{\sqrt{n}}(X_1 + \cdots + X_n)$

Is there a r.v. Z so that $\underbrace{Z_n \to Z \text{ a.s.}}_{*}$ (or in prob?)

Note CLT applies to Z_n's.

If * $Z_{2n} \to Z$

$Z_{2n} = \frac{1}{\sqrt{2n}}(X_1 + \cdots + X_n) + \frac{1}{\sqrt{2n}}(X_{n+1} + \cdots + X_{2n})$

$= \frac{1}{\sqrt{2}} Z_n + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{n}}(X_{n+1} + \cdots + X_n)$

$\sqrt{2}(Z_{2n} - \frac{1}{\sqrt{2}} Z_n) = \frac{1}{\sqrt{n}}(X_{n+1} + \cdots + X_{2n})$

RHS $= \frac{1}{\sqrt{n}}(X_{n+1} + \cdots + X_{2n}) \Rightarrow N(0,1)$

LHS $\to \sqrt{2}(Z - \frac{1}{\sqrt{2}} Z)$

$= (\sqrt{2} - 1) Z$

$\sim N(0, (\sqrt{2}-1)^2)$

Prop 10.2.1

Lemma: If $X_n \Rightarrow X$ and $Y_n \xrightarrow{P} 0$ then

$X_n + Y_n \Rightarrow X$

Proof: $F_n =$ df of $X_n + Y_n$, $F =$ d.f. of X

Need to show $F_n(x) \to F(x)$ for all F-continuity points x.

$F_n(x) = P(X_n + Y_n \le x) = P(X_n + Y_n \le x, |Y_n| \le \delta)$

$+ P(X_n + Y_n \le x, |Y_n| > \delta)$

$\le P(X_n \le x + \delta)$ (choose δ so $x+\delta$ is an F-continuity pt)

$+ P(|Y_n| > \delta)$

$\to F(x+\delta) + \lim P(|Y_n| > \delta)$

$= F(x+\delta) + 0$

$\le F(x) + \epsilon$ (choose δ for whatever $\epsilon > 0$ we chose)

Similarly, for $\delta > 0$, F cont. at x, we can choose $x - \delta$ to be F-continuity point

$F(x) - \epsilon \le F(x-\delta)$

$F(x-\delta) - \epsilon \le F_n(x-\delta)$ for all sufficiently large n

$\boxed{F_n(x-\delta) \to F(x-\delta)}$

$= P(X_n + Y_n \le x - \delta)$

$= P(X_n + Y_n \le x - \delta, |Y_n| \le \delta) + P(X_n + Y_n \le x - \delta, |Y_n| > \delta)$

$\le P(X_n + Y_n \le x - \delta + \delta) + P(|Y_n| > \delta)$

$\le P(X_n \le x - \delta + \delta) + P(|Y_n| > \delta)$

$= F_n(x) + P(|Y_n| > \delta)$

Given $\epsilon > 0$, $\exists n_0(x, \delta, \epsilon)$

$\ni \forall n \ge n_0$ $F_n(x-\delta) - F(x-\delta) \le \epsilon$

$-\epsilon \le F_n(x-\delta) - F(x-\delta) \le \epsilon + F(x-\delta)$

$F(x-\delta) - \epsilon \le F_n(x-\delta)$

Eg. $Z \sim N(0,1)$

$X_n = Z$

$X_n + Y_n \Rightarrow N(0,1)$

$E(X_n + Y_n)$

$E(Y) = a_n$

$Y_n = \begin{cases} a_n & \text{w.p. } \frac{1}{n} \\ 0 & \text{w.p. } 1 - \frac{1}{n} \end{cases}$ $a_n \to \infty$

since $Y_n \xrightarrow{P} 0$

For $\epsilon > 0$

$P(|Y_n| > \epsilon) = P(Y_n = a_n) = \frac{1}{n} \to 0$

$E(Z)$

)

$E(X_n + Y_n)$ $\qquad\qquad\qquad\qquad$ P(lim

$= 0 + E(Y_n) = \frac{a_n}{n}$

$E(X_n+Y_n) \Rightarrow 0 = E(Z)$

(1) $a_n = 1$ $\quad E(X_n+Y_n) \to 0 = E(Z)$

(2) $a_n = n$ $\quad E(X_n+Y_n) = \frac{n}{n} = 1 \not\to 0 = E(Z)$

(3) $a_n = n^2$ $\quad E(X_n+Y_n) = \frac{n^2}{n} = n \to \infty \neq 0 = E(Z)$

(4) Choose any period function and let $\frac{a_n}{n}$ run through these periods and get (for some of these choices)

$\liminf_{n\to\infty} E(X_n+Y_n) \neq \limsup_{n\to\infty} E(X_n+Y_n)$

so $E(X_n+Y_n)$ does not have a limit

$$Y_n = \begin{cases} a_n & \text{wp. } \frac{1}{2n} \\ 0 & \text{wp } 1-\frac{1}{n} \\ -a_n & \text{wp } \frac{1}{2n} \end{cases}$$

X_n, Y_n be independent

Find examples so

$X_n + Y_n \Rightarrow N(0,1)$

but $\mathrm{Var}(X_n+Y_n) \geq 1$

but can have limits 1, anything between $(1,\infty)$

limit ∞, or no limit

Now return to Theorem 10.1.1

Proof. (a) $\Rightarrow$ (b):

logical implication

$\mu_n(A) = \int I_A \, d\mu_n$ $\qquad$ $f^- \leq I_A \leq f^+$

$\mu(A) = \int I_A \, d\mu$

A interval

$0 \leq \int (f^+ - f^-) d\mu$

$\leq \int I_{(a-\epsilon, a+\epsilon)} d\mu$

$+ \int I_{(b-\epsilon, b+\epsilon)} d\mu$ $\qquad$ $\partial A = \{a\} \cup \{b\}$

$\mu(\partial A) = 0$ by Assump (b)

Also $(a-\epsilon, a+\epsilon) \downarrow \{a\}$ as $\epsilon \downarrow 0$.

$\therefore \mu((a-\epsilon, a+\epsilon)) \to 0$

$\therefore$ by (a)

for ex fixed

$\int f^- \, d\mu_n \leq \int I_A \, d\mu_n \leq \int f^+ \, d\mu_n$

$n \to \infty$ $\quad \downarrow \qquad\qquad\qquad\qquad \downarrow$

$\int I_A d\mu - \epsilon \leq \int f^- \, d\mu \leq \int I_A d\mu \leq \int f^+ \, d\mu \leq \int I_A \, d\mu + \epsilon$

still need to show for other measurable sets A.

---

(5) $\Rightarrow$ (1) ✓

(5) $\Rightarrow$ (2) ✓

(3) $\Rightarrow$ (4) [Skorokhod's Theorem]

(return to this next day)

Application of Skorokhod: ...

Lemma: (1) If $Y_n \to Y$ a.s., then $Y_n \to Y$ in prob.
and (2) If $Y_n \to Y$ in prob. then $Y_n \Rightarrow Y$

If $X_n \Rightarrow X$
and if f is continuous then $f(X_n) \Rightarrow f(X)$

By Skorokhod,
$\exists\, Y_n, Y$ on prob triple
(1) $\mathcal{L}(Y_n) = \mathcal{L}(X_n)$, $\mathcal{L}(X) = \mathcal{L}(Y)$
$Y_n \to Y$ a.s.

$\therefore \exists A$, $P(A) = 1$, $\omega \in A$, $Y_n(\omega) \to Y(\omega)$

$\therefore f(Y_n(\omega)) \to f(Y(\omega))$

$\therefore f(Y_n) \to f(Y)$ a.s., and hence in distribution

$\therefore P(f(X_n) \le x) = P(f(Y_n) \le x)$

$\to P(f(Y) \le x)$ for all x for which $P(f(Y) \le x)$ is cont. at x.

$= P(f(X) \le x)$