March 24, 2016 2:28 PM

Use Theorem 10.1.1 (c) as our (usual) definition of $\Rightarrow$ in $\mathbb{R}$

ie $\mu_n = \mathcal{L}(X_n)$, $\mu = \mathcal{L}(X)$

$\mu_n \Rightarrow \mu$ iff $\mu_n((-\infty, x]) \to \mu((-\infty, x])$ as $n \to \infty$

for all x s.t. $\mu(\{x\}) = 0$

$\mu_n((-\infty, x]) = F_n(x)$, F_n = cdf of $\mathcal{L}(X_n)$

F = cdf of $\mu = \mathcal{L}(X)$

$\mu(\{x\}) = 0$ iff F is continuous at x

Special case:

X_n, X are $\mathbb{N}_0$ valued, ie non-negative integer valued

$F_n(x) = P(X_n \le x)$

$= \sum_{j=0}^{[x]} P(X_n = j)$ ← is a finite sum (sum of a finite number of terms)

$\overset{?}{\to} F(x) = \sum_{j=0}^{[x]} P(X = j)$

at all x for which F is continuous

ie all x except for those that are point masses of $\mathcal{L}(X)$

ie $x \neq j$, $j \in \mathbb{N}_0$

$a_j = P(X = j)$

$a_{j,n} = P(X_n = j)$

$F_n(x) = \sum_{j=0}^{[x]} a_{j,n} \overset{?}{\to} F(x) = \sum_{j=0}^{[x]} a_j$

$\therefore F_n \Rightarrow F$ iff $\underset{a_{j,n}}{P(X_n = j)} \to \underset{a_j}{P(X = j)}$

Eg. $X_n \sim \text{Binom}(n, p_n)$, $np_n \to \lambda > 0$ as $n \to \infty$

To show X_n converges in distribution to Poisson, λ

show ($j \in \mathbb{N}_0$) $P(X_n = j) = \binom{n}{j} p_n^j (1-p_n)^{n-j}$ $\to \dfrac{\lambda^j e^{-\lambda}}{j!}$ as $n \to \infty$

$\binom{n}{j} = \dfrac{n^{(j)}}{j!} = \dfrac{n(n-1)\cdots(n-j+1)}{j!}$

Eg ②: $X_n \ge 0$, $X \ge 0$ w.p.1 (a.s.) X_n iid F

$Y_n = \min(X_1, \ldots, X_n)$

G_n = cdf of Y_n

For $x > 0$

$G_n(x) = P(Y_n \le x) = 1 - P(Y_n > x)$

$= 1 - (1 - F(x))^n$

If $x > 0$ $F(x) = \int_0^x f(u)\,du \doteq x \cdot f(0)$ (if $f(0) = \lim_{x \to 0^+} f(x)$)

x small, f pdf continuous, $\int_0^x f(u)\,du \doteq \int_0^x f(0)\,du$

$\frac{F(x)}{x} \to \lambda > 0$ as $x \to 0^+$

Aside: small x, say $\left(\frac{w}{n}\right)$

$G_n\left(\frac{w}{n}\right) = 1 - \left(1 - \frac{w}{n}\right)^n \to 1 - e^{-w}$

$= P\left(Y_n \le \frac{w}{n}\right)$

$= P(n \cdot Y_n \le w)$

$\equiv H_n(w)$ $H_n = \mathcal{L}(nY_n)$

$F(x)$ x small

$\doteq \lambda x^2$

$\equiv H_n(w)$ $\quad H_n$

$= \lambda x^2$

If $\frac{F(x)}{x^2} \to \lambda > 0$

$$G_n(\quad) = 1 - (1 - F(x))^n$$

$$G_n\left(\frac{w}{\sqrt{n}}\right) = 1 - \left(1 - \lambda\left(\frac{w}{\sqrt{n}}\right)^2\right)^n$$

$$H_n(w) = P(\sqrt{n}Y_n \le w) = 1 - \left(1 - \frac{\lambda w^2}{n}\right)^n \to 1 - e^{-\lambda w^2}$$

$H_n =$ dist. of $\sqrt{n}Y_n$

Cannot work directly using this method to study Central Limit Theorems.

$X_n \sim \text{Binom}(n, \theta)$ $\quad \theta$ fixed

$$Z_n = \frac{(X_n - n\theta)}{\sqrt{n}\sqrt{\theta(1-\theta)}}$$

$\theta = \frac{1}{2}$

$$Z_n = \frac{(X_n - \frac{n}{2})}{\frac{1}{2}\sqrt{n}}$$

$$F_n(x) = P(Z_n \le x)$$

$$= P\left(X_n \le \frac{n}{2} + \frac{1}{2}\sqrt{n} \cdot x\right)$$

$$= \sum_{j=0}^{\frac{n}{2} + \frac{1}{2}\sqrt{n}\,x} \binom{n}{j} \left(\frac{1}{2}\right)^j \left(\frac{1}{2}\right)^{n-j}$$

$$= \sum_{j=0}^{\frac{n}{2} + \frac{1}{2}\sqrt{n}\,x} \binom{n}{j} \cdot \frac{1}{2^n}$$

April 7 $\quad$ 12:30 – 3:30 EST (EDT?)

$\equiv$ 9:30 – 12:30 PST (PDT?)