March 28, 2016 1:59 PM

Can you hear us?
We did not hear you.

Characteristic Functions Chapter 11

MGF $X \sim \mu$
$M_X(t) = E(e^{tX}) < \infty$ for all $|t| \le t_0$ for some $t_0 > 0$
Implies $E(X^k)$ finite for all $k \in N$

Characteristic Functions:

$\varphi(t) = E(e^{itX})$ $\quad i = \sqrt{-1}$, $t \in \mathbb{R}$ $\quad \theta \in R$

$= E(\cos(tX) + i\sin(tX))$ $\quad e^{i\theta}$: complex exponential

$= E(\cos(tX)) + iE(\sin(tX))$ $\quad |e^{i\theta}| = \sqrt{Re(e^{i\theta})^2 + Im(e^{i\theta})^2} = 1$

$|\varphi(t)| \le E(|e^{itX}|)$
$\le E(1) = 1$

$|\varphi(t+h) - \varphi(t)| = |E(e^{i(t+h)X}) - E(e^{itX})|$
$= |E(e^{itX}(e^{ihX} - 1))|$
$\le E(|e^{itX}| \cdot |e^{ihX} - 1|)$
$\le \int |e^{ihx} - 1| \cdot \mu(dx)$ $\quad \mathcal{L}(X) = \mu$
$\to 0$ as $h \to 0$

$\therefore \varphi$ is uniformly continuous in t.

$|e^{itx} - 1|$
$\le |e^{itx}| + 1$
≤ 2

$e^{ihx} \to 1$ as $h \to 0$

Prop. 11.0.1
If X has finite k^{th} moment, has characteristic function φ, then for all $0 < j \le k$ (j integer)
$\varphi^{(j)}(t) = E((iX)^j e^{itX})$
and so $\varphi^{(j)}(0) = i^j E(X^j)$

Proof (outline)
$F_t = (iX)^{j-1} e^{itX}$
$F_t' = (iX)^j e^{itX}$
$|F_t'| \le |X|^j$, $\quad E(|X|^j) \le E(|X|^k) < \infty$

§11.1 Continuity Theorem

Inversion Formula
Theorem 11.1.1
μ is Borel prob. measure on R $\quad$ ($(\mathbb{R}, \mathcal{B}, \mu)$ prob triple)
$\varphi \quad \int \quad ^{itx} \quad (dx)$

μ is Borel prob. measure

$$\varphi(t) = \int_{\mathbb{R}} e^{itx}\,\mu(dx)$$

Then if $\mu(\{a\}) = \mu(\{b\}) = 0$

$$\mu([a,b]) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\,\varphi(t)\,dt$$

$$\overset{\shortparallel}{\mu((a,b))}$$

Lemma 11.1.2

$$\int_{\mathbb{R}}\int_{-T}^{T}\left|\frac{e^{-ita}-e^{-itb}}{it}\right| dt\,\mu(dx) \le 2T(b-a)$$

Lemma 11.1.3

For $T \ge 0$, $\theta \in \mathbb{R}$

$$\lim_{T\to\infty}\int_{-T}^{T}\frac{\sin(\theta t)}{t}\,dt = \pi\,\mathrm{sign}(\theta) = \pi\,\mathrm{sgn}(\theta)$$

Proof:

$$\int_{-T}^{T}\frac{\sin(\theta t)}{t}\,dt = \mathrm{sgn}(\theta)\cdot\int_{-T}^{T}\frac{\sin(|\theta| t)}{t}\,dt$$

$s = |\theta| t$

$t = \frac{s}{|\theta|}$

$dt = \frac{1}{|\theta|}\cdot ds$

$$= \mathrm{sgn}(\theta)\int_{-T|\theta|}^{T|\theta|}\frac{\sin(s)}{\frac{s}{\cancel{|\theta|}}}\cdot\frac{1}{\cancel{|\theta|}}\,ds$$

$$= \mathrm{sgn}(\theta)\int_{-T|\theta|}^{T|\theta|}\frac{\sin(s)}{s}\,ds$$

$$\lim_{T\to\infty}\int_{-T}^{T}\frac{\sin(\theta t)}{t}\,dt = \lim_{T\to\infty}\mathrm{sgn}(\theta)\int_{-T|\theta|}^{T|\theta|}\frac{\sin(s)}{s}\,ds$$

$$= 2\,\mathrm{sgn}(\theta)\lim_{T\to\infty}\int_{0}^{T|\theta|}\frac{\sin(s)}{s}\,ds$$

$$= 2\,\mathrm{sgn}(\theta)\cdot\int_0^\infty\frac{\sin(s)}{s}\,ds$$

$$\frac{1}{s} = \int_0^\infty e^{-us}\,du$$

$$\int_0^\infty \frac{\sin(s)}{s}\,ds = \int_0^\infty \sin(s)\int_0^\infty e^{-us}\,du\,ds$$

$$\overset{?}{=} \int_0^\infty\left\{\int_0^\infty \sin(s)\cdot e^{-us}\,ds\right\}du$$

(Fubini's Theorem p 110; one of the sufficient conditions in this theorem apply; $\sin(s)e^{-us} \ge -1 > -\infty$)

$$I_u = \int_0^\infty \sin(s)e^{-us}\,ds$$

(integrate by parts

$dv = e^{-us}$ $\quad dv = -ue^{-us}\,ds$

$dw = \sin(s)\,ds$

$w = -\cos(s)$

$d(vw) = v\,dw + w\,dv$

$$= (-\cos(s))\cdot e^{-us}\Big|_0^\infty + (-u)(-1)\int \cos(s)e^{-us}\,ds$$

$$= -(-1) - u\int_0^\infty \cos(s)e^{-us}\,ds$$

$$= 1 - u\int_0^\infty \cos(s)e^{-us}\,ds$$

$dw = \cos(s)\,ds$

$w = \sin(s)$

$v = e^{-us}$

$dv = -ue^{-us}$

$$= 1 - u\left\{\sin(s)e^{-us}\Big|_0^\infty - (-u)\int_0^\infty \sin(s)e^{-us}\,ds\right\}$$

$$2 \quad \infty \quad 0 \quad)e^{-us}\,ds$$

$$-(-u)\int_0^\infty \sin(s)e^{\;\;}\,ds \Big|$$

$$= 1 - u^2 \int_0^\infty \sin(s) e^{-us}\,ds$$

$$= 1 - u^2 I_u$$

$$\therefore I_u \cdot (1+u^2) = 1 \qquad \therefore I_u = \frac{1}{1+u^2}$$

$$\therefore \int_0^\infty \frac{\sin(s)}{s}\,ds = \int_0^\infty I_u\,du = \int_0^\infty \frac{1}{1+u^2}\,du$$

$$= \arctan(u)\Big|_0^\infty = \frac{\pi}{2}$$

Return to Inversion Formula (Theorem 11.1.1)

$$\frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\,\varphi(t)\,dt$$

$$= \frac{1}{2\pi}\int_{-T}^{T}\int_{\mathbb{R}} \frac{e^{-ita} - e^{-itb}}{it} \cdot e^{itx}\,\mu(dx)\,dt$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{-T}^{T} \frac{e^{+it(x-a)} - e^{+it(x-b)}}{it} \cdot dt \cdot \mu(dx)$$

$$(*) \quad = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{-T}^{T}\left(\frac{\sin(t(x-a)) - \sin(t(x-b))}{t}\right)dt\,\mu(dx)$$

Aside:

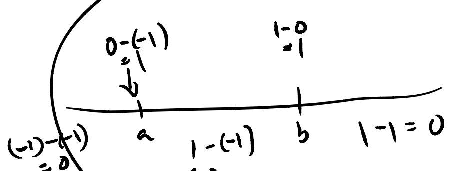
$$\frac{e^{it(x-a)} - e^{it(x-b)}}{it} = \underbrace{\left\{\frac{\cos(t(x-a)) - \cos(t(x-b))}{it}\right\}}_{\substack{\text{bounded odd function} \\ \text{so symmetric integral} \\ \text{(for fixed } T) = 0}} + i\,\frac{(\sin(t(x-a)) - \sin(t(x-b)))}{it}$$

$$\therefore (*) = \frac{1}{2\pi}\int_{\mathbb{R}} \Bigg\{\underbrace{\int_{-T}^{T} \frac{\sin(t(x-a)) - \sin(t(x-b))}{t}\,dt}_{\to \pi(\mathrm{sgn}(x-a) - \mathrm{sgn}(x-b))}\Bigg\}\mu(dx)$$

$$\therefore \lim_{T\to\infty} (*) = \frac{1}{2\pi}\int_{\mathbb{R}}\left\{\lim_{T\to\infty}\int_{-T}^{T}\frac{\sin(t(x-a)) - \sin(t(x-b))}{t}\,dt\right\}\mu(dx)$$

(dominated convergence theorem)

$$= \frac{1}{2}\int_{\mathbb{R}}\left\{\mathrm{sgn}(x-a) - \mathrm{sgn}(x-b)\right\}\mu(dx)$$



$$\frac{1}{2}\left(\mu(\{a\}) + 2\mu((a,b)) + \mu(\{b\})\right)$$

$$= \mu((a,b)) + \frac{1}{2}\mu(\{a\}) + \frac{1}{2}\mu(\{b\})$$

Corollary ...

If X, Y r.v's, ch. functions φ_X, φ_Y,
then $\mathcal{L}aw(X) = \mathcal{L}aw(Y)$ iff $\varphi_X = \varphi_Y$

Helly Selection Principle Lemma 11.1.8

Suppose $\{F_n\}$ is a sequence of cdf's on $\mathbb{R}$ (prob measures μ_n)
ie $\mu_n((-\infty, x]) = F_n(x)$, prob measures μ_n
Then $\exists$ subsequence $n_k \uparrow$ and a function F,
F non-decreasing, right continuous, such that
$F_{n_k}(x) \to F(x)$ as $k \to \infty$, for all x such that
F is continuous at x

Proof:

Fix $q_1 \in Q$ = set of rationals

$\{F_n(q_1)\}$ $\exists$ $n_{k,1} \uparrow \infty$ $F_{n_{k,1}}(q_1) \to G(q_1)$

$q_2 \neq q_1 \in Q$. $\exists$ $n_{k,2} \in \{n_{j,1} \mid j \geq 1\}$

$n_{k,2} \uparrow \infty$, $F_{n_{k,2}}(q_2) \to G(q_2)$

Continue, ...

Take n_ℓ = ℓ^{th} element of ℓ^{th} subsequence

$F_{n_\ell}(q_j) \to G(q_j)$ as $\ell \to \infty$,
and for any $q_j \in Q$

$G \uparrow$ in q (trivial)

$F(x) = \inf_{q \in Q} (G(q) : q \geq x)$

This F is right continuous, $F \uparrow$

If F is continuous at x

see top of p130.

Aside: $\mu_n = \frac{1}{2}\delta_{-n} + \frac{1}{2}\delta_n$

$F_n(x) = \mu_n((-\infty, x]) \to \frac{1}{2} \equiv F(x)$