Tightness (p 130)
A family of prob measures $\{\mu_n : n \geq 1\}$ is said to be tight iff $\forall \epsilon > 0$ $\exists$ $A = [a,b]$, such that
for all n, $\mu_n([a,b]) \geq 1 - \epsilon$

Theorem 11.1.10
If $\{\mu_n\}$ is a tight sequence of prob measures then
$\exists$ subsequence n_k, such $\mu_{n_k} \Rightarrow \mu$ for some prob measure μ.

Pf: $F_n(x) = \mu_n((-\infty, x]) \Rightarrow F$ along some subsequence (Helly)
$\therefore$ since $\{\mu_n\}$ is tight
$\{\mu_{n_k}\}$ (n_k subsequence from Helly)
is also tight.

$\therefore \lim_{x \to \infty} F(x) \stackrel{?}{=} 1$

Given $\epsilon > 0$, $\exists$ $[a,b]$ $\forall n$
$\Rightarrow \mu_n([a,b]) \geq 1-\epsilon$

$F_{n_k}(b) - F_{n_k}(a) \geq 1-\epsilon \longrightarrow$ choose a, b F-cont. points
$\therefore \lim F_{n_k}(a) = F(a)$
$\lim F_{n_k}(b) = F(b)$

$\therefore \lim_{x \to \infty} F(x) - \lim_{x \to -\infty} F(x)$
$\geq F(b) - F(a)$
$= \lim_{k \to \infty} (F_{n_k}(b) - F_{n_k}(a)) \geq 1 - \epsilon$

Corollary 11.1.11

Lemma 11.1.13
$\{\mu_n\}$ sequence of prob measures
characteristic functions $\phi_n(t) = \int_{\mathbb{R}} e^{itx} \mu_n(dx)$
Suppose $\exists$ g, such that
$\phi_n(t) \to g(t)$ for each t
and g is continuous at 0.
Then $\{\mu_n\}$ is tight

Pf.
$\mu_n\left((-\infty, -\tfrac{2}{y}] \cup [\tfrac{2}{y}, \infty)\right) = \int_{|x| \geq \frac{2}{y}} 1 \, \mu_n(dx)$ $\quad (1 - \frac{1}{y|x|}) \geq \frac{1}{2}$

$\leq 2 \int_{|x| \geq \frac{2}{y}} \left(1 - \frac{1}{y|x|}\right) \mu_n(dx)$

$\leq 2 \int_{|x| \geq \frac{2}{y}} \left(1 - \underbrace{\frac{\sin(yx)}{yx}}\right) \mu_n(dx)$

$= \int_{|x| \geq \frac{2}{y}} \frac{1}{y} \int_{-y}^{y} (1 - \cos(tx)) \, dt \, \mu_n(dx)$

$= \int \frac{1}{y} \int_{y}^{y} \left(1 - (\cos(tx) + i \sin(tx))\right) dt \cdot \mu_n(dx)$

$(1 - e^{itx}) \mu_n(dx) \, dt$

$= \frac{1}{y} \int^{y} \int$

$$= \int_{x \in \cancel{\mathbb{R}} \; \cancel{|x| \ge \frac{2}{y}}} \frac{1}{y} \int_{-y}^{y} \left(1 - \left(\cos(tx) \ldots \right) \right)$$

$$= \frac{1}{y} \int_{-y}^{y} \int_{\cancel{|x| \ge \frac{2}{y}} \; x \in \mathbb{R}} \left(1 - e^{itx}\right) \mu_n(dx)\, dt$$

$$= \frac{1}{y} \int_{-y}^{y} \int_{x \in \mathbb{R}} \left(1 - e^{itx}\right) \mu_n(dx)\, dt$$

$$\le \frac{1}{y} \int_{-y}^{y} \left(1 - \varphi_n(t)\right) dt$$

$\varphi_n(t) \to g(t)$, g cont at 0

$\therefore \forall \epsilon > 0 \; \exists \delta \ni \; |g(t) - \underset{1}{\underset{\shortparallel}{g(0)}}| \le \epsilon$ for all $|t - 0| \le \delta$

Take y_0 , $0 < y_0 < \delta$

$$\therefore \mu_n\left(\left(-\infty, -\tfrac{2}{y_0}\right] \cup \left[\tfrac{2}{y_0}, \infty\right) \right)$$

$$\le \frac{1}{y_0} \int_{-y_0}^{y_0} |1 - \varphi_n(t)|\, dt$$

$$\le \frac{1}{y_0} \cdot 2y_0 \cdot \epsilon = 2\epsilon$$

<u>Continuity Theorem</u>