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Continuity Theorem

Theorem 11.1.14 (Cont. Theorem) (on $\mathbb{R}$)

$\{\mu_n\}$ sequence of prob measures, characteristic functions ϕ_n. μ measure, ϕ = ch. function of μ. Then $\mu_n \Rightarrow \mu$ iff $\phi_n(t) \to \phi(t)$ for all $t \in \mathbb{R}$

CLT (Theorem 11.2.2)

$Z \sim N(0,1)$

$\varphi(t) = e^{-\frac{t^2}{2}} = M_Z(it)$

$M_Z = $ MGF of Z

X_j iid $E(X_j) = m$, $Var(X_j) = v$

$Z_n = \frac{\sqrt{n}(\bar{X}_n - m)}{\sqrt{v}} = \frac{\sum_{j=1}^n (X_j - m)}{\sqrt{nv}}$

If φ = ch. function of X_1

$$\varphi_n(t) = E(e^{itZ_n}) = E\left(e^{i\frac{t}{\sqrt{nv}}\sum_{j=1}^n (X_j - m)}\right)$$

$$= E\left(e^{\frac{it}{\sqrt{nv}}\sum_{j=1}^n X_j}\right) \cdot e^{-\frac{itnm}{\sqrt{nv}}}$$

$$= \varphi\left(\frac{t}{\sqrt{nv}}\right)^n \cdot e^{-\frac{it\sqrt{n}m}{\sqrt{v}}}$$

$$= \left(\varphi\left(\frac{t}{\sqrt{nv}}\right) \cdot e^{-\frac{itm}{\sqrt{nv}}}\right)^n$$

or

$$= \left(\psi\left(\frac{t}{\sqrt{n}}\right)\right)^n \quad \text{where } \psi(t) = E\left(e^{i\frac{(X_1 - m)}{\sqrt{v}}}\right)$$

$$= \left(1 - \frac{1}{2}\left(\frac{t}{\sqrt{n}}\right)^2 + o\left(\frac{1}{n}\right)\right)^n$$

$$\to e^{-\frac{t^2}{2}}$$

$\psi(0) = 1$

$\psi'(0) = 0$

$\psi''(0) = i^2 E\left(\left(\frac{X_1 - m}{\sqrt{v}}\right)^2\right) = -1$

$\Downarrow$

$P\left(\frac{1}{\sqrt{n}}\sum_{j=1}^n X_j \le x\right) - \Phi(x) \to 0$

See § 11.4 Method of Moments

Addition Results Needed in Statistics (and Stochastic Processes - eg Brownian Motion)

Need $\mathbb{R}^d$-valued r.v.s CLT

One can do this via characteristic function

$\underset{\sim}{X}$ $\mathbb{R}^d$ valued

$\underset{\sim}{t} \in \mathbb{R}^d$

$\varphi(\underset{\sim}{t}) = E(e^{i\langle \underset{\sim}{t}, \underset{\sim}{X}\rangle})$

Need

(1) Inversion formula

$d = 2$

$\mu([a_1, b_1] \times [a_2, b_2])$

$\mu\left(\bigtimes_{j=1}^d [a_j, b_j]\right)$

$$\int^{T_1} \int^{T_d} e^{-ita_j} \; e^{-itb_j} \; \varphi(\underset{\sim}{t})\, d\underset{\sim}{t}$$

$\mu\left(\bigtimes_{j=1}^{d}[a_j, b_j]\right)$

$$= \lim_{\underline{T}\to\infty} \frac{1}{(2\pi)^d}\int_{-T_1}^{T_1}\cdots\int_{-T_d}^{T_d} \frac{e^{-it_j a_j} - e^{-it_j b_j}}{it_j} \varphi(\underline{t})\, d\underline{t}$$

$= *$

in 1D: $\frac{1}{2}\mu(\partial[a,b]) + \mu((a,b))$ p 128

$= \mu((a,b))$ if $\mu(\partial[a,b]) = \mu(\{a,b\}) = 0$

If $\mu\left(\partial\left(\bigtimes_{j=1}^{d}[a_j, b_j]\right)\right) = 0$

$* = \mu\left(\partial\left(\bigtimes_{j=1}^{d}(a_j, b_j)\right)\right)$

Tool called Cramér-Wold Device

$\underset{\sim}{X}$ $\mathbb{R}^d$ valued.

Fix $\underline{a} \in \mathbb{R}^d$, $Y = \langle \underset{\sim}{a}, \underset{\sim}{X}\rangle = \underline{a}^T \underline{X} \in \mathbb{R}$

Now suppose $\underline{X}_j$, $j = 1, \dots, n$ are iid $\mathbb{R}^d$ valued,

$\underset{\sim}{\mu} = E(\underset{\sim}{X}_1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$A = \mathrm{Var}(\underline{X}_1) = E\left((\underline{X}_1 - \mu)(\underline{X}_1 - \mu)^T\right)$

$= E(\underline{X}_1 \underline{X}^T)$

$= \begin{pmatrix} \mathrm{Var}(X_{11}) & \mathrm{cov}(X_{11}, X_{12}) & \cdots \\ & \mathrm{Var}(X_{12}) & \\ & & \ddots \\ & & & \mathrm{Var}(X_{1d}) \end{pmatrix}$

$= \left(\mathrm{cov}(X_{1j}, X_{1\ell})\right)_{j,\ell = 1,\dots,d}$

Then $Y_j = \langle \underline{a}, \underline{X}_j\rangle$

has $E(Y_j) = 0$, $\mathrm{Var}(Y_j) = \underset{\sim}{a}^T A \underline{a}$

$\therefore$ 1D CLT applies

$Z_n = \frac{1}{\sqrt{n}}\sum_{j=1}^{n} Y_j \Rightarrow N(0, \underline{a}^T A \underline{a})$

$\parallel$

$\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\langle \underline{a}, \underline{X}_j\rangle = \langle \underline{a}, \underline{\underline{\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\underline{X}_j}}\rangle$

Multivariate Normal Characteristic Functions

$W \sim MVN(\underset{\sim}{0}, A)$

$\varphi(\underline{t}) = E\left(e^{i\langle \underline{t}, W\rangle}\right) = E\left(e^{i\underset{\sim}{t}^T W}\right)$

$= e^{-\frac{1}{2}\underline{t}^T A \underline{t}}$

$s\langle \underline{a}, Y_n\rangle$
$= s\cdot\sum a_j Y_{nj}$

$\phi_n(s) = E\left(e^{isY_n}\right) = E\left(e^{i\langle s\underset{\sim}{a}, X_n\rangle}\right)$

ch. function of Y_n

$\forall$, $E\left(e^{isZ_n}\right) \to e^{-\frac{1}{2}s^2\underline{a}^T A a}$

of Y_n

$$\varphi_n(s) = E(e^{isZ_n}) \to e^{-\frac{1}{2}s^2 \underset{\sim}{a}' A a)}$$

$$= e^{-\frac{1}{2}(s\underset{\sim}{a})^t A(s\underset{\sim}{a})} = E(e^{i(s\underset{\sim}{a})W})$$

$$E(e^{i\langle s\underset{\sim}{a}, \frac{1}{\sqrt{n}}\sum_{j=1}^{n} X_j\rangle}) \qquad W \sim MVN(\underset{\sim}{0}, \underset{\sim}{A})$$

ch. function of $\frac{1}{\sqrt{n}}\sum_{j=1}^{n} X_j$ evaluated at $s\underset{\sim}{a}$

Thus since $t \in \mathbb{R}$ is of the form $s\underset{\sim}{a}$, $\underset{\sim}{a} = \frac{t}{\|t\|}$, $s \in \mathbb{R}$

this Cramér-Wold device shows the $\mathbb{R}^d$ characteristic function of $\frac{1}{\sqrt{n}}\sum_{j=1}^{n} \underset{\sim}{X}_j$ converges the MVN$(0, A)$ ch. function.

eg: X_j, $j = 1, \dots, n$ iid $\mathbb{R}$ valued.

$$\hat{\mu}_{1n} = \frac{1}{n}\sum_{j=1}^{n} X_j$$

$$\hat{\mu}_{2n} = \frac{1}{n}\sum_{j=1}^{n} X_j^2$$

$$\hat{\mu}_{3n} = \frac{1}{n}\sum_{j=1}^{n} X_j^3$$

γ_n = sample skewness

$$= \frac{\hat{\mu}_{3n}}{\sqrt{\hat{\mu}_{2n} - \hat{\mu}_{1n}^2}} = g(\hat{\mu}_{1n}, \hat{\mu}_{2n}, \hat{\mu}_{3n})$$

To apply delta method we need CLT

$$\sqrt{n}\left(\begin{pmatrix}\hat{\mu}_{1n}\\ \hat{\mu}_{2n}\\ \hat{\mu}_{3n}\end{pmatrix} - \begin{pmatrix}\mu_1\\ \mu_2\\ \mu_3\end{pmatrix}\right) \qquad \mu_k = \text{population moments}$$

$$\underset{\sim}{Y}_j = \begin{pmatrix} X_j\\ X_j^2\\ X_j^3\end{pmatrix} \in \mathbb{R}^3 \text{ valued}$$

$$E(\underset{\sim}{Y}_j) = \begin{pmatrix}\mu_1\\ \mu_2\\ \mu_3\end{pmatrix}$$

$$\mathrm{Var}(\underset{\sim}{Y}_j) = \begin{pmatrix} \mathrm{cov}(X_1, X_1) & \mathrm{cov}(X_1, X_1^2) & \mathrm{cov}(X_1, X_1^3) \\ \mathrm{cov}(X_1^2, X_1) & \cdots & \\ \mathrm{cov}(X_1^3, X_1) & & \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{Var}(X_1) & E(X_1^3) - E(X_1)E(X_1^2) & \cdots \\ & & \\ & & \end{pmatrix}$$

$$= A$$

$\therefore$ By Cramér-Wold (in our case gives $\mathbb{R}^d$ CLT)

$$\sqrt{n}\begin{pmatrix}\hat{\mu}_{1n} - \mu_1\\ \hat{\mu}_{2n} - \mu_2\\ \hat{\mu}_{3n} - \mu_3\end{pmatrix} \Rightarrow MVN(\underset{\sim}{0}, A)$$

Return to Rosenthal

§ 11.3 Re: CLT $\qquad X_j$ iid $E(X_j)=0$, $Var(X_j)=1$

Berry-Esseen: $\qquad \Phi(x) = N(0,1)$ cdf at x

$$\sup_x \left| P\left(\frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_j \le x \right) - \Phi(x) \right| \le \frac{3\rho}{\sqrt{n}} \qquad = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

conditions: and $E(|X_1|^3) = \rho$

§ 11.4

Method of Moments

Theorem 11.4.1

μ_n sequence of prob measures

$$\int x^k \mu_n(dx) \to \int x^k \mu(dx)$$

If μ is determined by its moments then

$\mu_n \Rightarrow \mu$

$$\begin{aligned} \mu_n([-R,R]) &= P(|Y_n| \le R) \\ &= 1 - P(|Y_n| > R) \\ &\ge 1 - \frac{Var(Y_n)}{R^2} \\ &\to 1 - \frac{K_2}{R^2}, \quad K_2 = \lim \int x^2 \mu_n(dx) \\ &\qquad\qquad\qquad = \int x^2 \mu(dx) \end{aligned}$$

See Exercise 11.4.2